Final Examination, 12 Dec 2002 (Solutions)
SM311O (Fall 2002)

1. (a) Let $\mathbf{v} = \langle x^2 z, \sin(ay), z\sqrt{x}\rangle$ where a is a constant. Determine a so that the divergence of $\mathbf{v}$ vanishes at the point $P = (4, 0, 1)$.
   Solution: div $\mathbf{v} = 2xz + a\cos(ay) + \sqrt{x}$. At P the divergence has the value $10 + a$, which vanishes when $a = -10$.
   (b) Let $\mathbf{v} = \langle y^2 - x, y - x^2, 0\rangle$. Find the curl of $\mathbf{v}$. Is this flow irrotational anywhere?
   (c) Prove the identity $\nabla \times \nabla\phi = \mathbf{0}$ if ϕ is an arbitrary function of x, y, and z.

   Solution: $\nabla \times \nabla\phi = \nabla \times \langle \phi_x, \phi_y, \phi_z\rangle = \langle \phi_{zy} - \phi_{yz}, \phi_{xz} - \phi_{zx}, \phi_{yx} - \phi_{xy}\rangle = \langle 0, 0, 0\rangle$ because the order of differentiation does not matter for smooth (at least twice differentiable) functions.

2. Verify by direct differentiation if
   (a) $u(z) = e^{2z}\cos 2z$ is a solution of $u'' + au' + bu = 0$ for any pair (a, b).
   Solutions: With $u = e^{2z}\cos 2z$ the differential operator $u'' + au' + bu$ takes the form
   $$e^{2\,z}\,\left((2\,a + b)\,\cos(2\,z) - 2\,(4 + a)\,\sin(2\,z)\right).$$
   . This expression vanishes for ALL z if $2a + b = 0$ and $a + 4 = 0$, or if $a = -4$ and $b = 8$.
   (b) $u(x, y) = \sin 3x\cos 4y$ is an eigenfunction of the Laplace operator $-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$. What is the eigenvalue?
   Solution: A function u is an eigenfunction of $-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$ with eigenvalue λ if $-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = \lambda u$. Let $u = \sin 3x\cos 4y$. Then $-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 25\sin 3x\cos 4y = 25u$. So u is an eigenfunction with eigenvalue 25.

3. (a) Give a parametrization for the plane that passes through the points $(1, 1, 0)$, $(0, 2, 2)$, and $(3, 0, 3)$.
   Solution: First we use the three points on the plane and find two vectors that are parallel with the plane: $(1, 1, 0)$ and $(0, 2, 2)$ give $\mathbf{r}_1 = \langle -1, 1, 2\rangle$; $(0, 2, 2)$ and $(3, 0, 3)$ give $\mathbf{r}_2 = \langle 3, -2, 1\rangle$. Next we form the cross product of $\mathbf{r}_1$ and $\mathbf{r}_2$ to get $\mathbf{n}$ a normal vector to the plane:
   $$\mathbf{n} = \mathbf{r}_1 \times \mathbf{r}_2 = \langle 5, 7, -1\rangle.$$
   Let (x, y, z) be any point on the plane. Considering that $(1, 1, 0)$ is also on the plane, the vector $\mathbf{r}_3 = \langle x - 1, y - 1, z\rangle$ is parallel with the plane. Then $\mathbf{n}\cdot\mathbf{r}_3 = 0$ i.e. $5x + 7y - z = 12$.
   (b) Find a unit normal vector to the surface of the upper hemisphere of the Earth at the point whose longitude and latitude are 45 and 60 degrees, respectively.
   Solution: The upper hemisphere is parametrized as
   $$\mathbf{r}(u, v) = R\langle \cos u\cos v, \sin u\cos v, \sin v\rangle$$
   where R is the Earth's radius and u and v are longitude and latitude. Then $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_\mathbf{v}$ is normal to the hemisphere. Now $\mathbf{r}_u \times \mathbf{r}_\mathbf{v} = \langle R^2\,\cos(u)\cos(v)^2, R^2\cos(v)^2\,\sin(u), R^2\,\cos(v)\,\sin(v)\rangle$. The

magnitude of this vector is $R^2 \cos v$. Dividing $\mathbf{n}$ by its magnitude yields the desired unit vector $\mathbf{N} = \langle \cos(u)\cos(v), \cos(v)\sin(u), \sin(v)\rangle$. Finally evaluating at $u = \frac{\pi}{4}$ and $v = \frac{\pi}{3}$ gives $\mathbf{N} = \langle \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{\sqrt{3}}{2}\rangle$.

4. (a) The function $\phi(x,y) = ax^2y^2 - by^2 + ax - by$ is the potential for a velocity vector field $\mathbf{v}$. Determine all values of a and b so that the velocity of the particle located at $(2,-1)$ is $\langle 1, 2\rangle$.
   Solution: $\mathbf{v} = \nabla\phi = \langle 2axy^2 + a, 2ax^2y - 2by - b\rangle$. Evaluating $\mathbf{v}$ at $(2,-1)$ yields $\langle 5a, -8a+b\rangle$ which equals $\langle 1, 2\rangle$ if $a = \frac{1}{2}$ and $b = \frac{18}{5}$.
   (b) The function $\psi(x,y) = ax^2 + xy + by^2$ is the stream function of a velocity field $\mathbf{v}$. Find all a and b so that the velocity of the particle located at $(1,-2)$ has magnitude $\frac{1}{2}$.
   Solution: $\mathbf{v} = \langle \frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}\rangle = \langle x + 2by, -2ax - y\rangle$. Evaluating this vector at $(1,-2)$ and setting its magnitude equal to $\frac{1}{2}$ yields the equation $(2 - 2\,a)^2 + (1 - 4\,b)^2 = \frac{1}{4}$ for the set of all (a,b)s.

5. (a) Consider the velocity field $\mathbf{v} = (x^2z - x)\mathbf{k}$. Determine the flux of this fluid through the following two surfaces:
   i. a disk of radius 1 in the xy-plane and centered at the origin.
   Solution: First parametrize the disk: $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, 0\rangle$. Next compute $\mathbf{r}_u \times \mathbf{r}_v = \langle 0, 0, u\rangle$. Then $\int\int_S \mathbf{v}\cdot d\mathbf{r} = \int_0^{2\pi}\int_0^1 \mathbf{r}|_S \cdot \mathbf{r}_u \times \mathbf{r}_v\, dudv = \int_0^{2\pi}\int_0^1 \langle 0, 0, -u\cos v\rangle \cdot \langle 0,0,u\rangle\, dudv = 0$ so as much fluid is passing through S from below to above it as in the opposite direction.
   ii. a disk of radius 1 in the plane $z = 3$ and centered at the origin.
   Solution: First parametrize the disk: $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, 3\rangle$. Next compute $\mathbf{r}_u \times \mathbf{r}_v = \langle 0, 0, u\rangle$. Then $\int\int_S \mathbf{v}\cdot d\mathbf{r} = \int_0^{2\pi}\int_0^1 \mathbf{r}|_S \cdot \mathbf{r}_u \times \mathbf{r}_v\, dudv = \int_0^{2\pi}\int_0^1 \langle 0, 0, -u\cos v + 3u^2\cos^2 v\rangle \cdot \langle 0,0,u\rangle\, dudv = \frac{3\pi}{4}$.
   (b) Use the Stokes Theorem or compute the appropriate surface integral to determine the flux of vorticity of $\mathbf{v} = x^2\mathbf{k}$ through the surface of the upper hemisphere of a sphere of radius 2 centered at the origin.
   Solution 1) Direct computation: $\omega = \nabla\times\mathbf{v} = \langle 0, -2x, 0\rangle$. $\mathbf{r}(u,v) = \langle 2\cos u\cos v, 2\sin u\cos v, 2\sin v\rangle$. $\mathbf{r}_u \times \mathbf{r}_v = \langle 4\,\cos u\cos^2 v, 4\,\cos^2 v\sin u, 2\,\sin 2v\rangle$. Then $\int\int_S \omega\cdot d\mathbf{A} = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \langle 0, -4\cos u\cos v, 0\rangle \cdot \langle 4\,\cos u\,\cos^2 v, 4\,\cos^2 v\,\sin u, 2\,\sin 2\,v\rangle\, dvdu = 0$.
   2) Using the Stokes Theorem: We need to compute the line integral $\int_C \mathbf{v}\cdot d\mathbf{r}$. $\mathbf{r} = \langle 2\cos t, 2\sin t, 0\rangle$. But $\mathbf{v}|_C = \langle 0, 0, 4\cos^2 t\rangle$ is orthogonal to $\mathbf{r}'(t) = \langle -2\sin t, 2\cos t, 0\rangle$. So the line integral will vanish.

6. Consider the following wave equation initial-boundary value problem:

$$u_{tt} = 4u_{xx}, \qquad u(0,t) = u(3,t) = 0, \qquad u(x,0) = x(3-x), \quad u_t(x,0) = 0.$$

   (a) Use separation of variables and find the solution to this problem. Clearly indicate the process of separation of variables and the Fourier Series method used in obtaining this solution.
   Solution: From separation of variables we get that

$$u(x,t) = \sum_{n=1}^{\infty}(a_n \cos\frac{2n\pi t}{3} + b_n \sin\frac{2n\pi t}{3}) \sin\frac{n\pi x}{3}.$$

But $u_t(x,0) = 0$ which implies $b_n = 0$ for all n. $u(x,0) = x(3-x)$ so

$$a_n = \frac{(x(3-x), \sin\frac{n\pi x}{3})}{(\sin\frac{n\pi x}{3}, \sin\frac{n\pi x}{3})} = \frac{2}{3}\left(\frac{54}{n^3\,\pi^3} - \frac{54\,\cos(n\,\pi)}{n^3\,\pi^3}\right).$$

So

$$u(x,t) = \frac{2}{3}\sum_{n=1}^{\infty}\left(\frac{54}{n^3\,\pi^3} - \frac{54\,\cos(n\,\pi)}{n^3\,\pi^3}\right)\cos\frac{2n\pi t}{3}\sin\frac{n\pi x}{3}.$$

(b) Use the first nonzero term of the above solution and estimate $u(3/2, 3/4)$.
Solution: With $n=1$ the solution takes the form

$$u(x,t) = \frac{72}{\pi^2}\cos\frac{2\pi t}{3}\sin\frac{\pi x}{3}.$$

So $u(3/2, 3/4) = 0$.

(c) Use the first nonzero term of the above solution and estimate how long it takes for the wave to go though one complete vibration.
Solution: The period of $\cos\frac{2\pi t}{3}$ is $2\pi/(2\pi/3) = 3$.

7. Let $\mathbf{v} = \langle x^2+y^2, 2xy\rangle$ be the velocity field of a fluid. Compute the acceleration $\mathbf{a}$ of this flow. Does $\mathbf{a}$ have a potential p? If yes, find it.

Solution: $\mathbf{a} = \mathbf{v}\cdot\nabla v = \langle 2(x^3+3xy^2), 2(3x^2y+y^3)\rangle$. This vector has zero curl so there is a function p such that $\nabla p = \langle 2(x^3+3xy^2), 2(3x^2y+y^3)\rangle$. From

$$\frac{\partial p}{\partial x} = 2(x^3+3xy^2), \quad \frac{\partial p}{\partial y} = 2(3x^2y+y^3),$$

we get that $p = \frac{x^4}{2} + 3x^2y^2 + \frac{y^4}{2}$.

8. Let $\mathbf{\Omega}$ stand for the angular velocity of our planet.

(a) Noting that our planet rotates once every 24 hours, compute Ω where $\mathbf{\Omega} = \langle 0,0,\Omega\rangle$. What are the units of Ω?
Solution: $\Omega = \dfrac{2\pi\,\text{radians}}{(24\,\text{hours})(60\,\text{minutes/hour})(60\,\text{seconds/minute})} = 0.00007\,\text{rad/s}.$

(b) Use this value of $\mathbf{\Omega}$ and estimate the values in the centripetal acceleration $\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r})$ where $\mathbf{r}$ is the position vector to a typical point on the surface of the Earth. Assume that the radius of the Earth is 6000 kilometers.
Solution: Let $P = (x,y,z)$ be a point on the planet. Note that $\mathbf{\Omega} = \langle 0,0,\Omega\rangle$. Then $\mathbf{r} = \langle x,y,z\rangle$ and $\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}) = -\Omega^2\langle x,y,0\rangle$. x or y take their largest values at the equator which could be as large as 6000 kilometers. So the nonzero components of the centripetal acceleration could be as large as $0.00007^2\,(\text{rad/s})^2 \times 6000000\,\text{meters} = 0.0294\,\text{m/s}^2$, considerably smaller than $9.8\,\text{m/s}^2$ from the acceleartion of gravity.

9. Consider an incompressible fluid occupying the basin

$$D = \{(x, y, z)|0 \le z \le H\}.$$

Let $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be the velocity field of a motion generated in D. Suppose that we have been able to determine that

$$v_1(x, y, z) = 3x^2y^2 - x, \quad v_2(x, y, z) = yz,$$

but have only succeeded in measuring v_3 at the bottom of the basin and that this value is

$$v_3(x, y, 0) = x + y.$$

Determine v_3 everywhere in D. (Hint: What does incompressibility mean **mathematically**?)

Solution: From the equation of incompressibility we have

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 0$$

or

$$\frac{\partial v_3}{\partial z} = -\frac{\partial v_1}{\partial x} - \frac{\partial v_2}{\partial y}.$$

Substituting the values of v_1 and v_2 in the above relation yields

$$\frac{\partial v_3}{\partial z} = -2xy^2.$$

Integrating this result with respect to z from 0 to z and using the value of v_3 at $z = 0$ yields

$$v_3 = -2xy^2z + x + y.$$

Solution: Since the flow is incompressible,

10. A flow is called geostrophic if the velocity $\mathbf{v} = \langle u(x, y), v(x, y) \rangle$ and the pressure gradient ∇p are related by

$$(*) \qquad -fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \quad fu = -\frac{1}{\rho}\frac{\partial p}{\partial y},$$

where ρ, a constant, is the density of the fluid, and f is the coriolis parameter.

(a) Assuming that f is constant, prove that the divergence of $\mathbf{v}$ must vanish.

Solution: From the equations of motion we have

$$u = -\frac{1}{\rho f}\frac{\partial p}{\partial y}, \quad v = \frac{1}{\rho f}\frac{\partial p}{\partial x}.$$

Now div $\mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}$ which is equal to

$$-\frac{1}{\rho f}\frac{\partial^2 p}{\partial x \partial y} + \frac{1}{\rho f}\frac{\partial^2 p}{\partial y \partial x} = 0$$

(b) Prove that the particle paths of a geostrophic flow and its isobars coincide.

Solution: Note that $\mathbf{v} \cdot \nabla p = -\frac{1}{\rho f} \frac{\partial p}{\partial y} \frac{\partial p}{\partial x} + \frac{1}{\rho f} \frac{\partial p}{\partial x} \frac{\partial p}{\partial y} = 0$. So $\mathbf{v}$ and ∇p are orthogonal. Since ∇p is orthogonal to isobars, and since $\mathbf{v}$ is tangential to particle paths, particle paths and isobars coincide.

(c) Consider a high pressure field in a geostrophic flow in the northern hemisphere (where $f > 0$). By appealing to the equations in (*) explain whether this high pressure field results in a clockwise or a counterclockwise motion.

Solution: Without loss of generality, assume that the high pressure occurs at the origin of the coordinate system. Let P be a point in the first quadrant. Then ∇p at P points toward the origin because 0 is a maximum of p. Then $\frac{\partial p}{\partial x} \leq 0$ and $\frac{\partial p}{\partial y} \leq 0$ at P (draw a picture to convince yourself of this). Going back to the geostrophic equations, $u \geq 0$ and $v \leq 0$ at P which indicates that the motion is clockwise.